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Entropy of Sonic Black Hole in the Brick Wall Approach

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Abstract We calculate the free energy and the entropy of a massive scalar field in a sonic black hole by using the brick wall method. The leading order entropy turns out to be precisely in agreement with the entropy-area law.

Keywords Black hole entropy · Brick wall model · Sonic black hole

1 Introduction

Initiated by the original work of Bekenstein [1, 2], black hole entropy was studied in different approaches. The brick wall model which was developed by 't Hooft [3] can relate the Bekenstein-Hawking entropy S_{BH} to the statistical mechanical entropy S_{SE} of the thermal atmosphere of quantum fields near the black hole event horizon. In the brick wall model, there are divergences due to the number of modes close to the event horizon and 't Hooft [3] find that the leading order divergences can be regulated by introducing a cutoff parameter h_c, namely "the brick wall". Subsequently, the brick wall scenario has been studied intensively by many authors [4–28]. It was found that the brick wall model was efficacious for various black holes such as Schwarzschild-like black holes [26, 27], Kerr-Newman-like black holes [19, 20, 28], dilaton black holes [29], etc. One can also extending this method to arbitrary spin fields [15] and N-dimensional black holes [14] ($N > 4$).

In this paper we calculate the entropy of a sonic black hole by using the brick wall model. The sonic analog of a gravitational black hole was originally studied by Unruh [30, 31] who hope to perform experimental tests on black hole evaporation. By mapping certain aspects of black holes to supersonic flows one find that phononic propagation in a fluid is described by a wave equation which can be interpreted as fields propagation in an effective relativistic curved spacetime background [30–33]. The spacetime metric of the sonic analog black hole can be entirely determined by the physical properties of the fluid's density and flow velocity.

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The connection between supersonic flows and a black hole is so surprising that has attracted much attention.

Experimental consideration of sonic analog black holes includes many systems, such as atomic Bose-Einstein condensates [33–36], 1D Fermi-degenerate noninteracting gas [37], and one-dimensional transonic flows [38]. Similar to classical black holes, sonic black holes also have the structure of trapped regions, apparent horizon, event horizon, and Hawking radiation [33, 39]. A natural step is to study the entropy of a sonic black holes using the brick model which relates the black hole entropy to the thermal atmosphere of quantum fields near the black hole event horizon.

2 Sonic Black Holes

In this section we briefly review the sonic black hole which was first introduced by Unruh [30, 31]. If the fluid is inviscid and the flow of the fluid is irrotational, then the equation of motion for the velocity potential of a sound wave is identical to that for a minimally coupled massless scalar field propagating in a four dimensional black hole spacetime

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi)=0, \tag{1}$$

where ψ is the fluctuation of the velocity potential, which is interpreted as a sonic wave function, and the sonic metric $g_{\mu\nu}$ is given by [30, 31]

$$\begin{pmatrix} -(c^2-v_0^2) & -v_0^j \\ -v_0^i & \delta_{ij} \end{pmatrix}, \quad i,j=1,2,3, \tag{2}$$

where $c^{-2}=\frac{\partial\rho}{\partial p}$ is the local velocity of sound and v_0 is the leading order of the fluid's velocity. Assuming that the background flow is a spherically symmetric, stationary, and convergent flow, the metric of the sonic black hole can be written as [30, 31, 33]

$$ds^2=-c^2dt^2+(dr-v_0dt)^2+r^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{3}$$

If we introduce a coordinate transformation

$$d\tau=dt-\frac{v_0}{c^2-v_0^2}dr, \tag{4}$$

and assume that incompressibility and spherical symmetry, as ρ is position independent, which implies $v\propto 1/r^2$, we can set [33]:

$$v_0=c\frac{r_0^2}{r^2}, \tag{5}$$

where r_0 is the normalization constant. Setting $c=1$ by which we assume the local velocity of sound is a constant, we can rewrite this metric as

$$ds^2=-\left(1-\frac{r_0^4}{r^4}\right)d\tau^2+\left(1-\frac{r_0^4}{r^4}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{6}$$

We can see that the horizon of this sonic black hole is located at $r = r_0$. The inverse of the local temperature β can be defined as [40]

$$\frac{2\pi}{\beta} = \left.\frac{\partial_\tau g_{\tau\tau}}{2\sqrt{g_{\tau\tau} g_{rr}}}\right|_{r=r_0+\epsilon}. \tag{7}$$

From (7) we can easily get the Hawking temperature of the sonic black hole

$$T_H = \frac{1}{\pi r_0}, \tag{8}$$

and the black hole mass is given by

$$M = \frac{1}{8\pi}\oint_{r_0} \xi^{\mu;\nu} dS_{\mu\nu} = 2r_0 \tag{9}$$

where $\xi^\mu = (1, 0, 0, 0)$ is the Killing vector.

3 Free Energy and Entropy of a Sonic Black Hole

We now turn to calculate the free energy and the entropy of the sonic black hole using the brick wall model. Consider a massive scalar field in the background of the sonic black hole, following 't Hooft [3], we introduce a cutoff on the wave function outside the horizon

$$\Phi(x) = 0, \tag{10}$$

at

$$r \le r_0 + \epsilon, \quad r \ge L. \tag{11}$$

Here ϵ is a small positive quantity which signifies an ultraviolet cutoff, $L \gg r$ is an infrared cutoff.

The free energy for this system can be given by

$$F = -\int_0^\infty dE \frac{\mathrm{g}(E)}{e^{\beta E} - 1} \tag{12}$$

where $\mathrm{g}(E)$ is the number of states for a given E. In order to calculate the free energy of a scalar field we should find first the number of the states.

For convenience we rewrite the metric of the sonic black hole as

$$ds^2 = -\Delta(r)dt^2 + \Delta^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{13}$$

where

$$\Delta(r) = \left(1 - \frac{r_0^4}{r^4}\right). \tag{14}$$

The field equation for the massive scalar field can be given by

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - m^2\Phi = 0. \tag{15}$$

We take the WKB approximation

$$\Phi(t,r,\theta,\varphi)=\exp(-iEt)Y_{lm}(\theta,\phi)f_{E,l}(r). \tag{16}$$

The radial function is

$$f_{E,l}(r)\sim\exp\biggl(\pm i\int^r drk_{E,l}(r)\biggr), \tag{17}$$

where the radial momentum $k_{E,l}$ can be given as

$$k^2_{E,l}=\frac{(E^2-\Delta(r)(\frac{l(l+1)}{r^2}+m^2))}{\Delta^2(r)}. \tag{18}$$

By using the semi-classical quantization condition the non-negative integer number of radial modes $n_{E,l}$ in the brick wall is given by

$$\pi n_{E,l}=\int_{r_0+\epsilon}^{L} drk_{E,l}(r), \tag{19}$$

and the total number of the states with an energy less than E is given by

$$\mathrm{g}(E)=\Sigma_{modes}n_{E,l}\simeq\int^{l_{max}} dl(2l+1)\frac{1}{\pi}\int_{r_0+\epsilon}^{L} drk_{E,l}(r). \tag{20}$$

We can deduce the free energy as

$$\begin{aligned}F&=-\int_0^\infty dE\frac{\mathrm{g}(E)}{e^{\beta E}-1}\\&=-\frac{1}{\pi}\int^{l_{max}} dl(2l+1)\int dE\frac{1}{e^{\beta E}-1}\times\int_{r_0+\epsilon}^{L} drk_{E,l}(r).\end{aligned} \tag{21}$$

The l integral can be performed explicitly, then we can obtain

$$\begin{aligned}F&=-\frac{2}{3\pi}\int\frac{dE}{e^{\beta E}-1}\int_{r_0+\epsilon}^{L}\frac{r^2}{\Delta^2}[E^2-m^2\Delta]^{3/2}\\&=-\frac{2}{3\pi}\int\frac{dE}{e^{\beta E}-1}\mathcal{I}(E).\end{aligned} \tag{22}$$

For a non-extremal black hole, we can expand Δ near the event horizon by Taylor series

$$\Delta=\Delta'_{r_0}(r-r_0)+\frac{1}{2}\Delta''_{r_0}(r-r_0)^2+\mathcal{O}(r-r_0)^3, \tag{23}$$

where $\Delta'_{r_0}=4/r_0$, $\Delta''_{r_0}=4/r_0^2$. Other quantities can be expanded similarly, then we can calculate $\mathcal{I}(E)$ close to the event horizon as

$$\begin{aligned}\mathcal{I}(E)&=\int_{r_0+\epsilon}^{L}\frac{E^3r_0^2}{\Delta'_{r_0}(r-r_0)^2}\biggl\{1+(r-r_0)\biggl[\frac{2}{r_0}-\frac{\Delta''_{r_0}}{\Delta'_{r_0}}-\frac{3}{2}\frac{\Delta'_{r_0}m^2}{E^2}\biggr]\biggr\}+\mathcal{O}(1)\\&=\frac{E^3r_{r_0}^2}{(\Delta'_{r_0})^2}\biggl\{\frac{1}{\epsilon}-\biggl[\frac{2}{r_0}-\frac{\Delta''_{r_0}}{\Delta'_{r_0}}-\frac{\Delta'_{r_0}m^2}{E^2}\biggr]\log\biggl(\frac{L}{\epsilon}\biggr)\biggr\}+\mathcal{O}(1).\end{aligned} \tag{24}$$

Using the following formulae:

$$\begin{aligned}\int_0^\infty dE\frac{E^3}{e^{\beta E}-1}&=\frac{\pi^4}{15\beta^4},\\ \int_0^\infty dE\frac{E}{e^{\beta E}-1}&=\frac{\pi^2}{6\beta^2},\end{aligned}\tag{25}$$

we can complete the E integral; the free energy can be given by

$$F=-\frac{2}{3\pi}\frac{r_0^2}{(\Delta'_{r_0})^2}\left\{\frac{1}{\epsilon}\frac{\pi^4}{15\beta^4}-\left[\frac{2}{r_0}-\frac{\Delta''_{r_0}}{\Delta'_{r_0}}\right]\frac{\pi^4}{15\beta^4}\log(\epsilon)+\frac{\pi^2m^2\Delta'_{r_0}}{4\beta^2}\log\left(\frac{L}{\epsilon}\right)\right\}+\mathcal{O}(1).\tag{26}$$

The entropy can be derived from the free energy

$$S=\beta^2\frac{\partial F}{\partial\beta}.\tag{27}$$

Notice that $\beta=\pi r_0$, we can obtain the entropy near the horizon as

$$S=\frac{1}{\epsilon}\frac{r_0^2\Delta'_{r_0}}{360}-\left[\frac{r_0^2}{360}\left(\frac{2\Delta'_{r_0}}{r_0}-\Delta''_{r_0}\right)-\frac{r_0^2m^2}{12}\right]\log\left(\frac{L}{\epsilon}\right)+\mathcal{O}(1).\tag{28}$$

The proper distance [3] of the brick wall from the horizon is related to the ultraviolet cutoff ϵ is given by

$$\begin{aligned}h_c&=\int_{r_0}^{r_0+\epsilon}\Delta^{-1/2}(r)dr\\ &=2\sqrt{\frac{\epsilon}{\Delta'_{r_0}}}+\mathcal{O}(\epsilon^{3/2}).\end{aligned}\tag{29}$$

Substituting (29) to (28) and noticing that $\Delta'_{r_0}=4/r_0$, $\Delta''_{r_0}=4/r_0^2$, we finally obtain the entropy of the sonic black hole

$$\begin{aligned}S_{BW}&\simeq\frac{r_0^2}{90h_c^2}-\left[\frac{r_0^2}{180}\left(\frac{2\Delta'_{r_0}}{r_0}-\Delta''_{r_0}\right)-\frac{r_0^2m^2}{6}\right]\log\left(\frac{L\times r_0}{h_c^2}\right)\\ &\simeq\frac{r_0^2}{90h_c^2}+\left[\frac{1}{90}-\frac{r_0^2m^2}{6}\right]\log\left(\frac{r_0^2}{h_c^2}\right)\quad(\text{when } L\to r_0).\end{aligned}\tag{30}$$

Here S_{BW} denotes the entropy of the brick wall. The leading order of (30) is the standard result deduced by 't Hooft [3], which is regarded as the well known Bekenstein-Hawking entropy $S_{BH}=\mathcal{A}_H/4=\pi r_0^2$, and the subleading order are the quantum corrections to the entropy of the black hole. If we restrict ourselves to the leading order of the entropy S_{BW}, from (8, 9) and $S_{BH}=\pi r_0^2$ we can easily see that the sonic black hole mass (M), temperature (T) and entropy (S_{BH}) satisfy the first law of black hole thermodynamics

$$dM=TdS=2dr_0.\tag{31}$$

4 Conclusion

In summary, we have studied the statistical entropy of a four dimensional sonic black hole using the brick wall model. The leading order entropy turns out to be precisely in agreement with the entropy-area law, the logarithmic terms are divergent due to the infinite number of modes close to the event horizon, we regulate these terms by introducing a cutoff h_c which is a proper distance away from the event horizon. Here we should acknowledge that the metric (6) is a very idealized and oversimplified one for general sonic black holes. For a rotating BTZ-like sonic black hole the superradiant modes should be also considered due to the energy conditions [41], but in the present paper we do not need to consider the superradiant modes.

However the renormalization of the divergence can also be achieved by using various schemes such as Pauli-Villars regularization [5, 19, 20] and conical singularity [20, 42–44] method which we do not address here.

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